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HYPERBOLICITY OF STEADY-STATE EQUATIONS OF GAS SHEAR FLOWS IN A THIN LAYER

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The steady-state three-dimensional motion of an ideal gas in a thin layer of variable height is considered. In the long-wave approximation, the equations of gas dynamics reduce to a system of integrodifferential equations. The generalized characteristics and hyperbolicity conditions of the obtained system are found.

Key words: *steady-state gas dynamics, integrodifferential equations, generalized characteristics.*

1. System of Equations. We consider the steady-state three-dimensional motion of a non-heat-conducting inviscid gas. The system of gas-dynamic equations has the form

$$uu_x + vu_y + wu_z + \rho^{-1}p_x = 0,$$

$$uv_x + vv_y + wv_z + \rho^{-1}p_y = 0, \qquad uw_x + vw_y + ww_z + \rho^{-1}p_z = 0,$$

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0, \qquad uS_x + vS_y + wS_z = 0.$$

Here u, v, and w are the velocity components, p is the pressure, ρ is the density, and S is the specific entropy. The system is closed by the equation of state

$$\rho = R(p, S).$$

Let us consider the gas flow between two rigid walls $z = 0$ and $z = h(x, y)$, on which the nonpenetration boundary conditions are satisfied:

$$w\Big|_{z=0} = 0, \qquad w\Big|_{z=h} = uh_x + vh_y.$$

Let L_0 and U_0 be the characteristic horizontal scale and flow velocity, H_0 is the vertical scale, and R_0 and S_0 are the characteristic density and entropy.

The substitution of variables

$$x = L_0x', \qquad y = L_0y', \qquad z = H_0z',$$

$$u = U_0u', \quad v = U_0v', \quad w = (U_0H_0/L_0)w', \quad \rho = R_0\rho', \quad p = R_0U_0^2p', \quad S = S_0S'$$

changes only the third momentum equation, which takes the following form (primes are omitted):

$$\varepsilon^2(uw_x + vw_y + ww_z) + \rho^{-1}p_z = 0$$

($\varepsilon = H_0/L_0$). The long-wave approximation is related to the assumption that $H_0 \ll L_0$ ($\varepsilon \ll 1$). In the limiting case $\varepsilon \to 0$, the pressure does not depend on the vertical coordinate:

$$p = p(x, y).$$

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The remaining equations are simplified by introducing the Lagrangian coordinate $\lambda \in [0,1]$, which parametrizes the material surfaces from $z = 0$ to $z = h$ [1]. Let us introduce the variable λ by the formula

$$z = \Phi(x, y, \lambda), \qquad \lambda \in [0,1],$$

where the function Φ is a solution of the initial-boundary-value problem

$$u(x, y, \Phi)\Phi_x + v(x, y, \Phi)\Phi_y = w(x, y, \Phi),$$

$$\Phi\Big|_{x=x_0} = \Phi_0(y, \lambda), \qquad \Phi\Big|_{\lambda=0} = 0, \qquad \Phi\Big|_{\lambda=1} = h.$$

It is assumed that $u(x_0, y, \Phi_0) \neq 0$ and $\Phi_0\Big|_{\lambda=0} = 0$, $\Phi_0\Big|_{\lambda=1} = h$.

Then, the long-wave equations become

$$uu_x + vu_y + \rho^{-1}p_x = 0, \qquad uv_x + vv_y + \rho^{-1}p_y = 0,$$

$$p_\lambda = 0, \qquad (uH)_x + (vH)_y = 0, \qquad uS_x + vS_y = 0,$$

where the new function $H(x, y, \lambda) = \rho\Phi_\lambda$ is introduced.

Taking into account that $h = \int\limits_0^1 \Phi_\lambda \, d\lambda$, we obtain the relation between the pressure and the functions H and S:

$$h(x, y) = \int\limits_0^1 \frac{H}{R(p, S)} \, d\lambda.$$

Differentiating this relation, we have

$$\nabla p = \Big(\int\limits_0^1 \frac{HR_p}{R^2} \, d\lambda \Big)^{-1} \Big(\int\limits_0^1 \frac{\nabla H}{R} \, d\lambda - \int\limits_0^1 \frac{HR_S}{R^2} \nabla S \, d\lambda - \nabla h \Big).$$

Here the operator ∇ is calculated from the variables x and y.

As a result, we arrive at the integrodifferential system of equations

$$A\boldsymbol{U}_x + B\boldsymbol{U}_y = \boldsymbol{G}, \tag{1.1}$$

where $\boldsymbol{U} = (u, v, H, S)^{\mathrm{t}}$, $\boldsymbol{G} = (\sigma h_x, \sigma h_y, 0, 0)^{\mathrm{t}}$, $\sigma = \Big(\int\limits_0^1 R^{-2} H R_p \, d\lambda \Big)^{-1}$, and A and B are linear operators. The operators A and B act on the trial function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^{\mathrm{t}}$ by the rules

$$A\boldsymbol{\varphi} = \Big(u\varphi_1 + \frac{\sigma}{R} \int\limits_0^1 \frac{\varphi_3}{R} \, d\lambda - \frac{\sigma}{R} \int\limits_0^1 \frac{HR_S}{R^2} \varphi_4 \, d\lambda, u\varphi_2, u\varphi_3 + H\varphi_1, u\varphi_4 \Big)^{\mathrm{t}},$$

$$B\boldsymbol{\varphi} = \Big(v\varphi_1, v\varphi_2 + \frac{\sigma}{R} \int\limits_0^1 \frac{\varphi_3}{R} \, d\lambda - \frac{\sigma}{R} \int\limits_0^1 \frac{HR_S}{R^2} \varphi_4 \, d\lambda, v\varphi_3 + H\varphi_2, v\varphi_4 \Big)^{\mathrm{t}}.$$

The propagation of steady-state perturbations in a plane–parallel gas shear flow in a channel of variable cross section was studied in [2], simple waves in three-dimensional flows of a homogeneous fluid were studied in [3], and three-dimensional steady-state simple waves in barotropic fluid flows in [4].

2. Characteristics and Eigenfunctionals. In [1], the concept of the hyperbolicity of systems of partial differential equations is extended to the case of differential equations with operator coefficients.

Let $\mathcal{B}$ be a Banach space of the vector functions $\boldsymbol{\varphi}(\lambda)$ and the operators A and B act in the space $\mathcal{B}$. Then, the vector (ξ, η) on the plane (x, y) is called characteristic if the pair (ξ, η) is a solution of the eigenvalue problem

$$\langle \boldsymbol{F}, (\xi A + \eta B)\boldsymbol{\varphi} \rangle = 0, \tag{2.1}$$

where $\boldsymbol{F} \in \mathcal{B}'$ is the desired eigenvector functional; $\boldsymbol{\varphi} \in \mathcal{B}$ is an arbitrary trial vector function. Problem (2.1) is solved for each fixed point (x, y).

A curve on the plane (x, y) is called a characteristic of Eq. (1.1) if the direction of the normal to it at each point is parallel to the direction (ξ, η). The equality

$$\langle \boldsymbol{F}, A\boldsymbol{U}_x + B\boldsymbol{U}_y \rangle = \langle \boldsymbol{F}, \boldsymbol{G} \rangle \tag{2.2}$$

is called the relation on the characteristic.

System (1.1) is called a hyperbolic system [1] if problem (2.1) has only real roots ξ and η and the set of relations on the characteristics (2.2) is equivalent to system (1.1).

Next, we consider Eq. (2.1) at a certain fixed point (x, y). By virtue of the homogeneity of the equation for ξ, and η, it is possible to assume that $\xi^2 + \eta^2 = 1$ and to introduce the new unknown γ (the angle between the tangent to the characteristic and the x axis):

$$\xi = -\sin\gamma, \qquad \eta = \cos\gamma.$$

In addition, we make the change of variables on the hodograph plane

$$u = q\cos\vartheta, \qquad v = q\sin\vartheta.$$

By virtue of the independence of the trial functions φ_j, Eq. (2.1) is split into the following system of four equations:

$$\langle F_1, \varphi_1 q\sin(\vartheta-\gamma)\rangle - \sin\gamma\,\langle F_3, H\varphi_1\rangle = 0, \qquad \langle F_2, \varphi_2 q\sin(\vartheta-\gamma)\rangle + \cos\gamma\,\langle F_3, H\varphi_2\rangle = 0,$$

$$\langle F_3, \varphi_3 q\sin(\vartheta-\gamma)\rangle + \sigma\int_0^1 \frac{\varphi_3}{R}\,d\lambda \Big\langle -\sin\gamma\, F_1 + \cos\gamma\, F_2,\ \frac{1}{R}\Big\rangle = 0,$$

$$\langle F_4, \varphi_4 q\sin(\vartheta-\gamma)\rangle - \sigma\int_0^1 \frac{HR_S}{R^2}\,\varphi_4\,d\lambda \Big\langle -\sin\gamma\, F_1 + \cos\gamma\, F_2,\ \frac{1}{R}\Big\rangle = 0.$$

We consider the case where $\gamma \neq \vartheta(\lambda) + m\pi$ for all $\lambda \in [0,1]$ and $m \in \mathbb{Z}$. In this case, the trial function $\boldsymbol{\varphi}$ can be replaced by $\boldsymbol{\varphi}/(q\sin(\vartheta-\gamma))$. As a result, we obtain the following expression for the functionals:

$$\langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle = \int_0^1 \frac{H(-\varphi_1\sin\gamma + \varphi_2\cos\gamma)}{Rq^2\sin^2(\vartheta-\gamma)}\,d\lambda - \int_0^1 \frac{\varphi_3}{Rq\sin(\vartheta-\gamma)}\,d\lambda + \int_0^1 \frac{HR_S}{R^2}\,\frac{\varphi_4}{q\sin(\vartheta-\gamma)}\,d\lambda,$$

where the corresponding eigenvalue γ should satisfy the secular equation

$$\chi(\gamma) \equiv 1 - \sigma\int_0^1 \frac{H}{R^2q^2\sin^2(\vartheta-\gamma)}\,d\lambda = 0. \tag{2.3}$$

It is easy to verify that if $\gamma = \vartheta^\nu \equiv \vartheta\Big|_{\lambda=\nu}$, the functionals

$$\langle \boldsymbol{F}^{1\nu}, \boldsymbol{\varphi}\rangle = \varphi_1^\nu\cos\vartheta^\nu + \varphi_2^\nu\sin\vartheta^\nu,$$

$$\langle \boldsymbol{F}^{2\nu}, \boldsymbol{\varphi}\rangle = -(R\varphi_1)'\Big|_{\lambda=\nu}\sin\vartheta^\nu + (R\varphi_2)'\Big|_{\lambda=\nu}\cos\vartheta^\nu - \frac{R}{q\vartheta'\varphi_3}H\Big|_{\lambda=\nu}, \quad \langle \boldsymbol{F}^{3\nu}, \boldsymbol{\varphi}\rangle = \varphi_4,$$

$$\langle \boldsymbol{F}^{4\nu}, \boldsymbol{\varphi}\rangle = \langle F_0^\nu, -\varphi_1\sin\vartheta^\nu + \varphi_2\cos\vartheta^\nu\rangle - \sigma\int_0^1 \frac{\varphi_3}{Rq\sin(\vartheta-\vartheta^\nu)}\,d\lambda + \sigma\int_0^1 \frac{HR_S}{R^2}\,\frac{\varphi_4}{q\sin(\vartheta-\vartheta^\nu)}\,d\lambda$$

$$\Big[\langle F_0^\nu, \varphi\rangle = R^\nu\varphi^\nu + \sigma\int_0^1 \frac{H(R\varphi - R^\nu\varphi^\nu)}{R^2q^2\sin^2(\vartheta-\vartheta^\nu)}\,d\lambda\Big]$$

are eigenfunctionals. Here and below, the prime denotes partial derivatives with respect to λ and the superscript ν denotes the value of the function for $\lambda = \nu$.

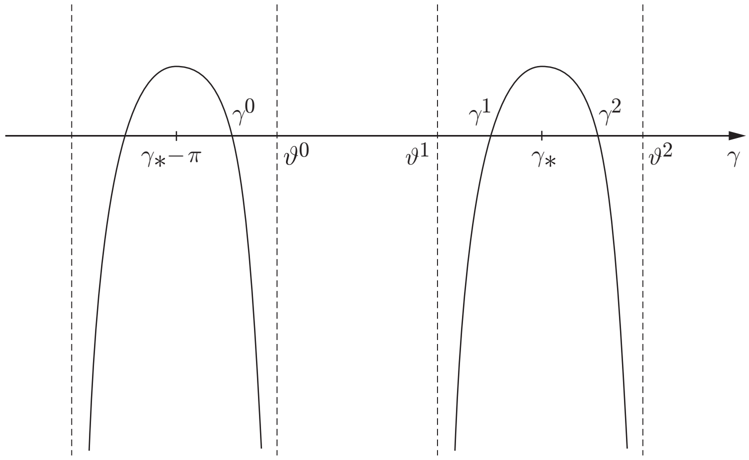

Fig. 1.

3. Condition for the Absence of Complex Roots. Thus, it is shown that the solutions of problem (2.1) is a set of values of ϑ^ν and γ^k, where $\vartheta^\nu = \vartheta\Big|_{\lambda=\nu}$ $[\nu \in (0,1)]$ and γ^k is a set of solutions (generally speaking, complex) of Eq. (2.3) that do not lie on the segment $[\min_\lambda \vartheta, \max_\lambda \vartheta]$. We now find the conditions under which Eq. (2.3) has only real roots γ.

We consider only the case where the function $\vartheta(\lambda)$ is monotonic. For definiteness, let $\partial\vartheta/\partial\lambda > 0$. We designate $\vartheta^0 = \vartheta\Big|_{\lambda=0}$ and $\vartheta^1 = \vartheta\Big|_{\lambda=1}$, where $\vartheta^1 - \vartheta^0 < \pi$ [otherwise, the function $\chi(\gamma)$ is not defined on the real axis].

Lemma 1. *For real γ, the function $\chi(\gamma)$ has the following properties:*

1) *$\chi(\gamma)$ is periodic with period π;*

2) *if $\sigma H/(R^2q^2\vartheta') \neq 0$ for $\lambda = 0$ and $\lambda = 1$, then $\chi \to -\infty$ for $\gamma \to \vartheta^1+0$ and $\gamma \to \vartheta^2-0$, where $\vartheta^2 = \vartheta^0+\pi$;*

3) *$\chi(\gamma)$ is convex (on the period).*

Indeed, the function $\sin^2$ is π-periodic; therefore property 1 is valid.

We note that if the values of γ differ by a magnitude that is a multiple of π, they determine the same characteristic normal.

Property 2 follows from the fact that the function $\chi(\gamma)$ can be written as

$$\chi(\gamma) = 1 + \frac{\sigma H}{R^2q^2\vartheta'} \cot\,(\vartheta-\gamma)\Big|_{\lambda=0}^{1} - \sigma\int_0^1 \Big(\frac{H}{R^2q^2\vartheta'}\Big)'\vartheta' \cot\,(\vartheta-\gamma)\,d\lambda.$$

Next, the second derivative has the form

$$\chi''(\gamma) = -2\sigma\int_0^1 \frac{H}{R^2q^2}\,\frac{1+2\cos^2(\vartheta-\gamma)}{\sin^4(\vartheta-\gamma)}\,d\lambda < 0,$$

whence follows property 3 of Lemma 1. The function $\chi(\gamma)$ is plotted schematically in Fig. 1. Lemma 1 leads to the following lemma.

Lemma 2. *In the interval $(\vartheta^1, \vartheta^2)$, the function $\chi(\gamma)$ reaches the single maximum $\chi(\gamma_*)$. In this case, if*

$$\chi(\gamma_*) > 0, \tag{3.1}$$

then the secular equation (2.3) has two different (with accuracy up to the period) real roots γ^1 and γ^2. At the points γ^1 and γ^2, the following inequalities hold:

$$\chi'(\gamma^1) > 0, \qquad \chi'(\gamma^2) < 0. \tag{3.2}$$

If $\chi(\gamma_*) < 0$, then Eq. (2.3) has no real roots.

Inequalities (3.2) follow from the fact that $\chi'' < 0$.

We now consider the function $\chi(\gamma)$ for the complex values of the variable γ. Because of the periodicity of $\chi(\gamma)$, it suffices to consider the strip $\operatorname{Re}\gamma \in (\gamma_* - \pi, \gamma_*)$ (Re denotes the real part of the complex number). We make the change of variables in the integral of the function χ (2.3):

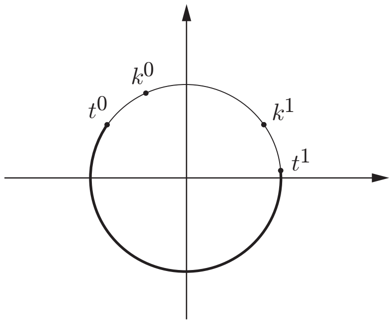

Fig. 2.

$$t = \mathrm{e}^{2i\vartheta}, \qquad k = \mathrm{e}^{2i\gamma}. \tag{3.3}$$

Then, on the arc Γ of the unit circle, t changes from $t^0 = \mathrm{e}^{2i\vartheta^0}$ to $t^1 = \mathrm{e}^{2i\vartheta^1}$, and $k \in \mathbb{C} \setminus \Gamma$. The sides of the strip $\operatorname{Re}\gamma = \gamma_* - \pi$ and $\operatorname{Re}\gamma = \gamma_*$ are mapped onto the sides of the cut along the ray $\arg k = 2i\gamma_*$, which can be sewed by virtue of the periodicity of the function $\chi(\gamma)$. The function $\chi(\gamma)$ can be brought to the form

$$\chi(k) = 1 - \frac{2i\sigma tH}{R^2q^2\vartheta'(k-t)}\Big|_{t^0}^{t^1} - \sigma \int\limits_\Gamma \Big(\frac{H}{R^2q^2\vartheta'}\Big)' \frac{1}{\vartheta'}\,\frac{dt}{t-k}.$$

We assume that the condition $\chi(\gamma_*) > 0$ is satisfied, i.e., that two real roots $\gamma^0 \in (\gamma_* - \pi, \vartheta^0)$ and $\gamma^1 \in (\vartheta^1, \gamma_*)$ exist. These roots correspond to the points $k^0 = \mathrm{e}^{2i\gamma^0}$ and $k^1 = \mathrm{e}^{2i\gamma^1}$ located on a unit circle of the complex plane. We also note that the complex roots of the function $\chi(\gamma)$ correspond to the roots of the function $\chi(k)$ that do not lie on the unit circle.

Let us consider a contour that consists of the following elements: the inner and outer sides of the cut along Γ, circles of small radius with centers at the points t^0 and t^1, and a circle of large radius with center at the coordinate origin (Fig. 2).

At the points $k = t^0$ and $k = t^1$, the function $\chi(k)$ has singularities of the type of a simple pole, $\chi \to 1$ as $k \to \infty$; therefore by virtue of the argument principle, the equation $\chi(k) = 0$ has only two roots k^0 and k^1, if

$$\chi^+ \neq 0, \quad \Delta \arg \chi^+/\chi^- = 0 \quad \text{along} \quad \Gamma. \tag{3.4}$$

4. Completeness of the System of Eigenfunctionals. Let conditions (3.1) and (3.4) and the equation $\langle \boldsymbol{F}^j, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{F}^{l\nu}, \boldsymbol{\varphi}\rangle = 0$ $[j = 1, 2;\ l = 1, 2, 3, 4;\ \nu \in (0, 1)]$ be satisfied. We prove that $\boldsymbol{\varphi} \equiv 0$.

From the equation $\langle \boldsymbol{F}^{3\nu}, \boldsymbol{\varphi}\rangle = 0$, we obtain $\varphi_4 \equiv 0$.

Let us introduce the function $\psi = -\varphi_1 \sin\vartheta + \varphi_2 \cos\vartheta$. From the equation $\langle \boldsymbol{F}^{1\nu}, \boldsymbol{\varphi}\rangle = 0$ it follows that $\varphi_1 = -\psi \sin\vartheta$ and $\varphi_2 = \psi\cos\vartheta$, and from the equation $\langle \boldsymbol{F}^{2\nu}, \boldsymbol{\varphi}\rangle = 0$ we obtain $\varphi_3 = H(R\psi)'/(Rq\vartheta')$.

Then, the equation $\langle \boldsymbol{F}^{4\nu}, \boldsymbol{\varphi}\rangle = 0$ can be brought to the form

$$R^\nu\psi^\nu + \sigma \int\limits_0^1 \frac{H(R\psi\cos(\vartheta - \vartheta^\nu) - R^\nu\psi^\nu)}{R^2q^2\sin^2(\vartheta - \vartheta^\nu)}\,d\lambda - \sigma\int\limits_0^1 \frac{H}{R^2q^2\vartheta'}\,\frac{(R\psi)'}{\sin(\vartheta - \vartheta^\nu)}\,d\lambda = 0.$$

It is easy to verify that this equation is satisfied for the functions $\psi = 1/(R\sin(\vartheta - \gamma^j))$ $(j = 1,\ 2)$. We set

$$\psi = \psi_0 + \frac{C_1}{R\sin(\vartheta - \gamma^1)} + \frac{C_2}{R\sin(\vartheta - \gamma^2)}, \tag{4.1}$$

where the constants C_1 and C_2 are chosen so that $\psi_0 = 0$ for $\lambda = 0, 1$. Then, the equation for ψ_0 can be brought to the form

$$\Big(1 - \sigma\int\limits_0^1 \Big(\frac{H}{R^2q^2\vartheta'}\Big)' \cot\,(\vartheta - \vartheta')\,d\lambda + \frac{\sigma H\cot\,(\vartheta - \vartheta^\nu)}{R^2q^2\vartheta'}\Big|_{\lambda=0}^1\Big) R^\nu\psi_0^\nu + \sigma\int\limits_0^1 \Big(\frac{H}{R^2q^2\vartheta'}\Big)' \frac{R\psi_0}{\sin(\vartheta - \vartheta^\nu)}\,d\lambda = 0.$$

Using the change of variables (3.3), we arrive at the equation

$$\Big(1 - \sigma\int\limits_\Gamma \Big(\frac{H}{R^2q^2\vartheta'}\Big)' \frac{1}{\vartheta'}\,\frac{dt}{t - t^\nu} + \frac{2i\sigma tH}{R^2q^2\vartheta'}\,\frac{1}{t - t^\nu}\Big|_{t^0}^{t^1}\Big)\frac{R^\nu\psi_0^\nu}{\sqrt{t^\nu}} + \sigma\int\limits_\Gamma \Big(\frac{H}{R^2q^2\vartheta'}\Big)' \frac{1}{\vartheta'}\,\frac{R\psi_0/\sqrt{t}}{t - t^\nu}\,dt = 0. \tag{4.2}$$

Equation (4.2) is a singular integral equation which is adjoint to the secular equation and has a Cauchy kernel for the function $R\psi_0/\sqrt{t}$.

We introduce the piecewise-analytical function

$$\Phi(z) = \frac{\sigma}{2\pi i} \int\limits_{\Gamma} \Big(\frac{H}{R^2 q^2 \vartheta'}\Big)' \frac{1}{\vartheta'} \, \frac{R\psi_0/\sqrt{t}}{t - z} \, dt = 0,$$

which is defined on a complex plane with a cut along Γ. According to the properties of the Cauchy integral (4.2), the function $\Phi(z)$ is bounded near the ends of the contour Γ and vanishes at infinity. By virtue of the Sokhotsky–Plemelj formulas and the properties of the Cauchy integral [5], the integral equation is transformed to the conjugation problem for the function $\Phi(z)$:

$$\Phi^+(t^\nu) = G(t^\nu)\Phi^-(t^\nu).$$

Here $G(t^\nu) = \chi^+(t^\nu)/\chi^-(t^\nu)$ is the coefficient of the conjugation problem. By virtue of conditions (3.4), the index of the conjugation problem is equal to zero; therefore, according to the general theory [5], the problem has only a trivial solution. From this, using the Sokhotsky–Plemelj formulas, we obtain $\psi_0 = 0$.

We note that $\langle \boldsymbol{F}^j, 1/(R\sin{(\vartheta - \gamma^k)})\rangle = 0$ for $j \neq k$ $(j, k = 1, 2)$; therefore, from (4.1) it follows that $C_j = 0$ because

$$\Big\langle \boldsymbol{F}^j, \frac{1}{R\sin{(\vartheta - \gamma^j)}}\Big\rangle = \chi'(\gamma^j) \neq 0.$$

The aforesaid leads to the following statement.

Statement 1. If conditions (3.1) and (3.4) are satisfied, Eqs. (1.1) are hyperbolic.

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